SECTION 10.3 (10.1): SERIES

Suppose we have a sequence $\{a_n\}_{n=1}^{\infty}$. What would it mean to 'add up' this sequence? Informally, we are asking of we can assign a value to the sum: $a_1 + a_2 + a_3 + \ldots$.

As usual in Calculus, we'll start with a pre-calculus notion (finite sums) and use limits to head off to infinity.

DEFINITION: Given a sequence $\{a_n\}_{n=1}^{\infty}$, the *n* **th partial sum**, denoted S_n, is the sum of the first n terms:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

Specifically, $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, and so on.

EXAMPLE 1: Consider the geometric sequence: $\left\{ \dfrac{9}{10^n} \right\}_{n=1}^{\infty}$

1. Compute the first five partial sums.

$$S_1 = a_1 = \frac{9}{10}$$

$$S_2 = a_1 + a_2 = \frac{9}{10} + \frac{9}{100} = \frac{99}{100}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} = \frac{999}{1000}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} = \frac{9999}{10000}$$

$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \frac{9}{100000} = \frac{99999}{100000}$$

2. Find a formula for the n th partial sum, S_n. Ans: $S_n = \dfrac{10^n - 1}{10^n} = 1 - \dfrac{1}{10^n}$

3. Find $\lim_{n \to \infty} S_n$. Ans: $\lim_{n \to \infty} S_n = 1$.

NOTE: The above reasoning shows $0.\overline{9} = 1$.

DEFINITION: Given a sequence $\{a_n\}_{n=1}^{\infty}$, we define the (infinite) **series**:

$$S = \sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} S_n,$$

provided the limit exists. If the limit exists, we say the series **converges**; otherwise, we say the series **diverges**.

NOTE 1: The value of the **series** S is the **limit** of the **sequence** of **partial** sums, $\{S_n\}_{n=1}^{\infty}$.

NOTE 2: Do you see the similarity between the definition of series and the definition of the improper integral:

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

Since series convergence is defined in terms of sequence convergence, the usual limit arithmetic applies. We summarize these properties below. Can you see which property of limits is being used in each case?

PROPERTIES OF CONVERGENT SERIES: Suppose c is a real number and that $\sum_{k=1}^{\infty} a_k = L$ and $\sum_{k=1}^{\infty} b_k = M$.

- **CONSTANT MULTIPLE RULE:** $\sum_{k=1}^{\infty} c\, a_k = c \sum_{k=1}^{\infty} a_k = c\, L$

- **SUM AND DIFFERENCE RULE:** $\sum_{k=1}^{\infty} (a_k \pm b_k) = \sum_{k=1}^{\infty} a_k \pm \sum_{k=1}^{\infty} b_k = L \pm K$

Geometric series play a vital role in our study of series. Since geometric series have a well-defined structure, we have well-developed formulas for these series.

FORMULAS FOR GEOMETRIC SERIES: Consider the geometric sequence: a, ar, ar^2, ar^3, ...

- If $r = 1$, then $S_n = \underbrace{a + a + a + \ldots + a}_{n \text{ times}} = n\, a$.

- If $r \neq 1$, then $S_n = a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a\,(1 - r^n)}{1 - r}$.

- If $a \neq 0$, the series $a + ar + ar^2 + ar^3 + \ldots$ converges if $|r| < 1$ and diverges otherwise.[1] Moreover, if $|r| < 1$:

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1 - r} = \frac{\text{the first term}}{1 - \text{the common ratio}}$$

EXAMPLE 2: Suppose you collect \$0.01 on April 1st, \$0.02 on April 2nd, \$0.04 on April 3rd, and so forth, doubling the amount of money you receive each day.

1. How much money will you collect on April 5th? April 15th?

   Ans: April 5th: \$0.16; April 15th: \$163.84.

2. What is the total amount of money you'll have collected as of April 5th? April 15th?

   Ans: Total as of April 5th: \$0.31; Total as of April 15th: \$327.67.

3. What is the total amount of money you'll have collected by the end of April?

   Ans: Total as of April 30th: \$10,737,418.23

[1] In other words, the series converges '**if and only if**' $|r| < 1$.

EXAMPLE 3: Determine if the following series converge or diverge. If the series converges, find its sum.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{3(-2)^k}{5^{k-1}}$ Ans: converges to $-\dfrac{30}{7}$.

2. $\displaystyle\sum_{k=1}^{\infty} \frac{3(-2)^{3k}}{5^{k-1}}$ Ans: diverges; $|r| = \dfrac{8}{5} > 1$.

3. $\displaystyle\sum_{k=2}^{\infty} \frac{4^k - 2^{k-1}}{3^{2k}}$ Ans: converges to $\dfrac{34}{105}$.

EXAMPLE 4: Consider the series:[2] $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}$

1. Determine all values x for which the series converges. Ans: $-2 < x < 2$.

2. What does the series converge to? Check your answer graphically. Ans: $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}} = \frac{1}{2-x}$

TELESCOPING SERIES: Not all series are geometric. Consider, for instance, the series: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2}$.
This series is not geometric (can you see why?) so we resort to looking at partial sums.

We get: $S_1 = \dfrac{1}{6}$, $S_2 = \dfrac{1}{4}$, $S_3 = \dfrac{3}{10}$, and $S_4 = \dfrac{1}{3}$. There doesn't seem to be a pattern evolving.

Taking a cue from integration, we employ partial fraction decomposition: $\dfrac{1}{k^2+3k+2} = \dfrac{1}{k+1} - \dfrac{1}{k+2}$.

Using this formulation, we get:

$$S_1 = \frac{1}{2} - \frac{1}{3}$$

$$S_2 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} - \frac{1}{4}$$

$$S_3 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2} - \frac{1}{5}$$

$$S_4 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{2} - \frac{1}{6}$$

A pattern now develops:[3] $S_n = \dfrac{1}{2} - \dfrac{1}{n+2}$ so that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}$.

[2] The convention with series is that the x^0 produced by the index of the sum is always 1 for convenience, even if $x = 0$.

[3] To **prove** this pattern holds for all n, we'd have to employ a technique called **mathematical induction**.

EXAMPLE 5: (VIDEO) Prove the series below coverge by finding their sum.

1. $\displaystyle\sum_{k=2}^{\infty} \frac{2}{4k^2-1}$ Ans: $\displaystyle\sum_{k=2}^{\infty} \frac{2}{4k^2-1} = \frac{1}{3}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+2k}$ Ans: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+2k} = \frac{3}{4}$.

Note that in this section, we could show series converge and actually determine what they converge to. This won't be the case as we move through the chapter. Indeed, the rest of the chapter is devoted to determining tests to determine whether or not a series converges - period. What a convergent series converges to will be a mystery - although, in some cases, we'll have tools to help us approximate the sums within a given error.

HOMEWORK: Section 10.3: 19 - 87 odd

APPENDIX: ANNUITIES

An important application of the geometric sum formula is the investment plan called an **annuity**. Annuities are investments where payments are deposited into the account on an on-going basis. Suppose you have an account with annual interest rate r which is compounded n times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. Suppose we wish to make ongoing deposits of P dollars at the *end* of each compounding period. Let A_k denote the amount in the account after k compounding periods. Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on A_1 so that our initial investment has grown to $A_1 + iA_1 = A_1(1+i) = P(1+i)$. We then add another payment, P at the end of the second period to get:[4]

$$A_2 = A_1(1+i) + P = P(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)$$

During the third compounding period, we earn interest on A_2 which then grows to $A_2(1+i)$. We add our third payment at the end of the third compounding period and factor to obtain:

$$A_3 = A_2(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)(1+i) + P = P(1+i)^2\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2}\right)$$

During the fourth compounding period, A_3 grows to $A_3(1+i)$, and when we add the fourth payment, and factor:

$$A_4 = P(1+i)^3\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3}\right)$$

This pattern continues so that at the end of the kth compounding, we get

$$A_k = P(1+i)^{k-1}\left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first k terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$.

Using the partial sum formula, we get:

$$1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{k-1}} = 1\left(\frac{1 - \dfrac{1}{(1+i)^k}}{1 - \dfrac{1}{1+i}}\right) = \frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1}\left(\frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}\right) = \frac{P\left((1+i)^k - 1\right)}{i}$$

If we let t be the number of years this investment strategy is followed, then $k = nt$, and we get:

THE FUTURE VALUE OF AN ORDINARY ANNUITY: Suppose an annuity offers an annual interest rate r compounded n times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit P is made at the end of each compounding period, the amount A in the account after t years is given by

$$A = \frac{P\left((1+i)^{nt} - 1\right)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into the equation above and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit P is made a the *beginning* of the compounding period instead of at the end, the annuity is called an **annuity-due**. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

[4]The reason for factoring out the $P(1+i)$ will become apparent soon.

EXAMPLE: An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the annuity in 30 years.

   We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

   $$A = \frac{50\left((1+0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

   Our final answer is \$50,225.75.

2. How many years will it take for the annuity to grow to \$100,000?

   To find how long it will take for the annuity to grow to \$100,000, we set $A = 100000$ and solve for t. We isolate the exponential and take natural logs of both sides of the equation.

   $$\begin{aligned} 100000 &= \frac{50\left((1+0.005)^{12t} - 1\right)}{0.005} \\ 10 &= (1.005)^{12t} - 1 \\ (1.005)^{12t} &= 11 \\ \ln\left((1.005)^{12t}\right) &= \ln(11) \\ 12t\ln(1.005) &= \ln(11) \\ t &= \tfrac{\ln(11)}{12\ln(1.005)} \approx 40.06 \end{aligned}$$

   This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering.